Week 6 (Midterm Review)
MATH 33A
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1. Let $v_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

   (a) Find the matrix representation of *a* linear transformation such that $\ker(T) = span\{v_1, v_2\}$.

   (b) Find the matrix representation of *a* linear transformation such that $Im(T) = span\{v_1, v_2\}$.

Solutions:

(a) In order to define a linear transformation, we need to define it on a basis. Since $\{v_1, v_2\}$ forms a linearly independent set, we can extend it to a basis by adding something else in, like $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for instance. So, consider $\{v_1, v_2, e_1\}$, which is a linearly independent set. We note that since $\mathbb{R}^3$ has dimension 3, $\{v_1, v_2, e_1\}$ forms a basis. Now, define $T : \mathbb{R}^3 \to \mathbb{R}$ such that $T(v_1) = T(v_2) = 0$ (since we need v_1, v_2 to span the kernel) and $T(e_1) = 1$ (this doesn't need to be 1, it could be anything nonzero. We can't have it be zero or else the kernel would be bigger than $span(v_1, v_2)$). From this, we note that given the matrix representation of T, M, we would have $M \begin{bmatrix} v_1 & v_2 & e_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, from which we can solve for $M = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & e_1 \end{bmatrix}^{-1}$

(b) Recall that given $A = \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}$, and $x = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, we have $Ax = \sum_{j=1}^{n} c_j A_j$.

As such, we can simply take $\begin{bmatrix} v_1 & v_2 \end{bmatrix}$ as our matrix.

(note that the answers given for both parts of this problem are far from unique!)

2. Let $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

   (a) Find the matrix representation of the projection onto v (ie. $T(x) = Proj_v(x)$).

   (b) Find the matrix representation of the reflection about v (ie. $T(x) = Refl_v(x)$).

   (c) Are any of these invertible? If so, compute their inverse.

Solutions

(a) By recalling the formula, we note that $Proj_v(e_1) = \frac{3}{25}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $Proj_v(e_2) = \frac{4}{25}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. These are the columns for the matrix representation.

(b) By recalling the formula, we note that $Refl_v(e_1) = \frac{1}{25}\begin{bmatrix} -7 \\ 24 \end{bmatrix}$ and $Refl_v(e_2) = \frac{1}{25}\begin{bmatrix} 24 \\ 7 \end{bmatrix}$. These are the columns for the matrix representation.

(c) The projection is not invertible, since it is collapsing a plane into a line (one may also show non-invertibility by row reducing). The reflection is invertible, since reflecting twice gives what we originally had (this can also be showed by squaring the matrix).

3. Invert the following matrix

$$\begin{bmatrix} 3 & 5 & 9 \\ 2 & 3 & 7 \\ 1 & 3 & 3 \end{bmatrix}.$$

Solution: Set up the augmented matrix $[A|I]$ and row reduce it to $[I|A^{-1}]$. You can do it!

4. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation such that $T(\begin{bmatrix}1\\1\\1\end{bmatrix}) = \begin{bmatrix}2\\3\\5\end{bmatrix}$, $T(\begin{bmatrix}1\\2\\3\end{bmatrix}) = \begin{bmatrix}5\\1\\2\end{bmatrix}$ and $T(\begin{bmatrix}0\\3\\1\end{bmatrix}) = \begin{bmatrix}1\\3\\3\end{bmatrix}$. Find the matrix representation of T. Is T invertible? If so, compute its inverse.

Solution: Let M be the matrix representation. Then we know that $M \begin{bmatrix}1 & 1 & 0\\1 & 2 & 3\\1 & 3 & 1\end{bmatrix} = \begin{bmatrix}2 & 5 & 1\\3 & 1 & 3\\5 & 2 & 3\end{bmatrix}$. It then follows that $M = \begin{bmatrix}1 & 1 & 0\\1 & 2 & 3\\1 & 3 & 1\end{bmatrix} \begin{bmatrix}2 & 5 & 1\\3 & 1 & 3\\5 & 2 & 3\end{bmatrix}^{-1}$.